Ripples in a wave tank.

\[ \text{Phase Velocity} \quad U_p = \frac{(2\pi S \lambda^{1/2})}{\lambda} \]

This has "physics" in it.

- \( U_p \) increases if \( S \) increases ... more surface tension means waves travel faster.
- \( U_p \) decreases if \( \rho \) increases ... if water gets more dense then waves travel slower.

However, in this case we are really interested in variation of \( U_p \) with \( \lambda \).

\[ U_p = \alpha \lambda^{1/2} = \alpha K^{1/2} \]

Some constant from above, another constant.

(a) \[ U_p = \alpha \lambda^{1/2} = \frac{\omega}{K} \]

This gives dispersion (If \( U_p \) is constant, no dispersion).

I will use \[ (K = \frac{2\pi}{\lambda}) \]

So phase velocity \[ U_p = \alpha K^{1/2} = \frac{\omega}{K} \]

\[ \therefore \omega = \alpha K^{3/2} \]

Group velocity \[ U_g = \frac{d\omega}{dK} = \frac{3}{2} \alpha K^{1/2} = \frac{3}{2} U_p \]
Ch 7 Q18/2.

(b) \( U_{\text{group}} = \frac{3U_{\text{phase}}}{2} \)

So \( U_{\text{group}} > U_{\text{phase}} \).

Crests on surface of water wave group (envelope) will move backwards.

(c) \( y(x,t) = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t) \)

Using trig identity

\[ = 2 \cos \left( \frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} t \right) \times \cos \left( \frac{k_1 - k_2}{2} x - \frac{\omega_1 - \omega_2}{2} t \right) \]

\[ = 2 \cos \left( k_{\text{avg}} x - \omega_{\text{avg}} t \right) \cos \left( k_{\text{mod}} x - \omega_{\text{mod}} t \right) \]

Consider a "snap-shot" at \( t = 0 \)

\[ = 2 \cos (k_{\text{avg}} x) \cos(k_{\text{mod}} x) \]

\[ k_{\text{avg}} = \frac{k_1 + k_2}{2} \]

\[ k_{\text{mod}} = \frac{k_1 - k_2}{2} \]

\[ \lambda_{\text{mod}} = \frac{2\pi}{k_{\text{mod}}} \]

Distance between peaks \( \frac{\lambda_{\text{mod}}}{2} \)
\[
K_1 = \frac{2\pi}{\lambda_1} = \frac{2\pi}{0.99} \text{ cm}^{-1}
\]

\[
K_2 = \frac{2\pi}{\lambda_2} = \frac{2\pi}{1.01} \text{ cm}^{-1}
\]

\[
K_{\text{mod}} = \frac{2\pi}{2} \left( \frac{\frac{1}{0.99} - \frac{1}{1.01}}{1.01 \times 0.99} \right)
\]

\[
\Rightarrow \lambda_{\text{mod}} = \frac{2\pi}{K_{\text{mod}}} \cdot \frac{2\pi}{2} \left( \frac{1.01 \times 0.99}{1.01 - 0.99} \right)
\]

\[
= 100 \text{ cm}
\]

\[
\therefore D_{\text{loop size}} = 50 \text{ cm}
\]
Appendix

Trig Identity

\[ 2 \cos \Theta \cos \phi = \cos (\Theta + \phi) + \cos (\Theta - \phi) \]

\[ \frac{\Theta}{2} = \frac{A+B}{2} \quad \Theta + \phi = A \]

\[ \phi = \frac{A-B}{2} \quad \Theta - \phi = B \]

\[ \Rightarrow 2 \cos \left( \frac{A+B}{2} \right) \cos \left( \frac{A-B}{2} \right) = \cos A + \cos B. \]
Ch 7 Q 21

(a) Take the example we studied of transverse waves

\[ L = (N+1) \ell \]

Modes:

- \( n=1 \):
  \[ \lambda = 2L \]

- \( n=2 \):
  \[ \lambda = L \]

- \( n=3 \):
  \[ \lambda = \frac{2L}{3} \]

etc.

You should be able to repeat this derivation.

See lecture VibWav L07f - page 18-19.

We have

\[ \omega = 2\omega_0 \sin \left( \frac{\pi n}{2(N+1)} \right) \]

\[ \omega_0 = \left( \frac{T_0}{Mc} \right)^{1/2} \]

\[ K = \frac{2\pi}{\lambda} \frac{n \ell}{L(N+1)} \]

\[ n = 1 \ldots N \]
Ch7 Q21\(\frac{1}{2}\).

(b) \(n^{th}\) mode

\[ \lambda_n = \frac{2(N+1)l}{n} \]

\[ K_n = \frac{\pi n}{(N+1) l} \]

\[ \omega_n = 2\omega_0 \sin \left( \frac{n\pi}{2(N+1)} \right) = 2\omega_0 \sin \left( \frac{K_n l}{2} \right) \]

**General**

\[ U_{\text{phase}} = \frac{\omega}{K} = \frac{(N+1)l2\omega_0 \sin \left( \frac{n\pi}{2(N+1)} \right)}{\pi n} = 2(N+1)\frac{\omega_0}{\pi n} \sin \left( \frac{K_n l}{2} \right) \]

\[ U_{\text{group}} = \frac{d\omega}{dk} = \frac{\omega_0 l}{\pi} \cos \left( \frac{K_n l}{2} \right) \]

**For \(n \ll N\)**

\[ U_{\text{phase}} = \frac{(N+1)l2\omega_0 \left( \frac{n\pi}{2(N+1)} \right)}{\pi n} \quad \text{since} \quad \sin \theta \approx \theta \quad \theta \ll 1. \]

\[ = l\omega_0 \]

\[ U_{\text{group}} = l\omega_0 \quad \text{since} \quad \cos \theta = 1 \quad \theta \ll 1. \]

Clearly \( U_{\text{phase}} \approx U_{\text{group}} \)
Cu7 Q21/3.

For $n = N+1$

$$k_{N+1} = \frac{\pi (N+1)}{(N+1) L} = \frac{\pi}{L}$$

$$U_{\text{phase}} = \frac{2w_0 \sin \left( \frac{k_{N+1} L}{2} \right)}{k_{N+1}}$$

$$\cos \frac{\pi}{2} = 0$$

$$U_{\text{group}} = w_0 L \cos \left( \frac{k_{N+1} L}{2} \right) = 0$$

But consider expressions we had for amplitudes of individual balls from L04 p15. Overall size of $n$th normal mode

$$A_{p,n} = C_n \sin \left( \frac{\pi n p}{N+1} \right)$$

$p$th ball in chain $n$th normal mode relative amplitude of each ball

For $n = N+1$ $A_{p,n+1} = C_n \sin \left( \frac{\pi (n+1) p}{(N+1)} \right) = 0$ for all $p$ integers.

The $n = N+1$ mode is identically zero, just as the $n = 0$ mode is also zero.
Note that the \( n = N+2 \) mode is just the same as the \( n = N \) mode, \( n = N+3 \) same as \( n = N-1 \) and so on.

\[
\sin \left( \frac{\pi n p}{N+1} \right) = \sin \left( \frac{\pi (N+2) p}{N+1} \right) = \sin \left( \frac{\pi p (2N+1)}{(N+1)} \right) = \sin \left( 2\pi \frac{p}{N+1} - \frac{\pi p N}{N+1} \right)
\]

Since \( p \) is an integer, this is simply \( \sin \left( 2\pi \frac{p}{N+1} \right) = \sin(-\theta) = -\sin(\theta) \)

\[
= -\sin \left( \frac{\pi p N}{N+1} \right)
\]

This is the same as expression for \( n = N \) mode except for global negative sign which is applied to all amplitudes and doesn't affect relative amplitudes (i.e., it just an arbitrary phase applied to all amplitudes).